Colin Petitjean (Univ. Bourgogne Franche-Comté)

On Lipschitz maps which attain their norm

Let X be a Banach space and M be a metric space equipped with a distinguished point denoted 0. We consider $Lip_0(M,X)$ the space of Lipschitz maps $f : M \to X$ which satisfy $f(0) = 0$. Equipped with the norm $\|f\|_L$, being the best Lipschitz constant of f, $Lip_0(M,X)$ is a Banach space. We then say that a Lipschitz map $f \in Lip_0(M,X)$ strongly attains its norm whenever there is $x \neq y \in M$ such that $\|f(x) - f(y)\|_X = \|f\|_L d(x,y)$.

Next, there is a different notion of norm attainment. It is known that there is a Banach space $\mathcal{F}(M)$ together with an isometry $\delta : M \to \mathcal{F}(M)$ such that every $f \in Lip_0(M,X)$ extends uniquely to a continuous operator $\overline{f} \in \mathcal{L}(\mathcal{F}(M),X)$ satisfying $\|\overline{f}\| = \|f\|_L$ and $\overline{f} \circ \delta = f$. The Banach space $\mathcal{F}(M)$ is the so called Lipschitz free space over M. We now say that a Lipschitz map $f \in Lip_0(M,X)$ attains its operator norm if there exists an element $\gamma \in \mathcal{F}(M)$ such that $\overline{f}(\gamma) = \|f\|_L$ and $\|\gamma\|_{\mathcal{F}(M)} = 1$.

We will analyze the relationships between the above-mentioned notions of norm attainment. This will lead us quite naturally to the study of the extremal structure of Lipschitz free spaces. In light of the celebrated Bishop–Phelps theorem, we will also analyze when the class of Lipschitz functions which strongly attain their norm is dense in $Lip_0(M,X)$.